(Tamtad)

The Hanging Chain Problem

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<u>Introduction – The Catenary Problem</u>

If you hold a length of chain by each end and let it hang freely in the middle, what mathematical shape does it make? Even without touching a chain, we might confidently say "It's a parabola!" Even Galileo, in his book *The Two New Sciences*, said "This chain will assume the form of a parabola" (Galilei et al. 149). Case closed. Thank you all for coming.

Perhaps we should prove this, however, since this is a mathematical inquiry? As we proceed, we must know that we walk a well-trodden path. This is a classic investigation called the Catenary Problem, derived from the Latin word catēna, meaning chain (Griva and Vanderbei). There are many real-life examples of hanging chains and ropes, but suspension bridges are perhaps the classic example. I recently visited a beautiful bridge called the Geierlay, roughly halfway between Frankfurt and Belgium. We shall determine an equation that describes the shape of all hanging chains, and specifically the Geierlay bridge.

The Physics and Math of Hanging Chains

This problem is hugely important in physics and engineering, but only a few physics concepts are required before we enter completely into the mathematical realm.

We start with a chain of mass M and length L; capitals are used to help us remember that they are the totals for the whole chain. The ends of the chain are fixed, and the chain is hanging motionless. Continuing to speak of it as a chain is useful to help us visualize it as a series of short straight sections, or links, but we will take the limit as these links become *very* short in order to differentiate and integrate along our chain. The solution, then, works equally well for a rope or cable.

Let us call the length of each link $d\ell$; its x and y components are dx and dy (Figure 1). These can all be related using the Pythagorean theorem:

$$d\ell^2 = dx^2 + dy^2$$

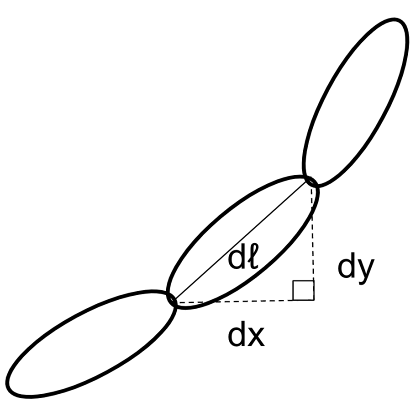

Figure 1: The length of one chain link, $d\ell$, and its x and y components, dx and dy (Self-made).

We then solve for $d\ell$ and factor out a dx from the right side which will simplify differentiating and integrating later:

$$d\ell = \sqrt{dx^2 + dy^2} = \sqrt{dx^2(1 + \frac{dy^2}{dx^2})} = dx\sqrt{1 + (\frac{dy}{dx})^2} = d\ell \quad \text{(Equation 1)}$$

Here is where physics comes into play: because the chain is at rest, the net forces must all equal zero. Newton's 2nd Law can be stated as $F = ma$, where F is the net force, m is mass, and a is acceleration. If the total forces do not equal zero then the chain would be accelerating and therefore moving. The only forces involved are those of gravity (F_g) and tension (T). These are all vector quantities, but for clarity we will use the x and y components separately and use signs to indicate their relative directions. Note that the notation $T_y(x + dx)$, for example, indicates the y-component of tension at point $x + dx$ as used in function notation rather than multiplication by $x + dx$.

The horizontal tensions are unaffected by gravity and there are no other horizontal forces, so to sum to zero they must be equal to a constant, but in opposite directions at the end of each chain link:

$$T_x(x) = C = -T_x(x + dx) \qquad \text{(Equation 2)}$$

Only one of these two equalities is necessary for our analysis, and the real constant C will appear in our final result. C can only be calculated for a particular catenary, so we shall later determine this and other constants for the Geierlay suspension bridge.

The vertical tensions are in opposite directions at opposite ends of each link, and their difference must equal and oppose the force of gravity in order to sum to zero:

$$T_y(x + dx) - T_y(x) = F_g \qquad \text{(Equation 3)}$$

If we stipulate that the chain is of uniform density, then the ratio of the mass of a single link, dm, to the total mass M is equal to the ratio of the length of a link, $d\ell$, to the total length L ("How Do You Find"): $\frac{dm}{M} = \frac{d\ell}{L}$ so then $dm = M\frac{d\ell}{L}$

Then using g as the acceleration of gravity on the Earth's surface, Newton's 2nd Law in this case becomes $F = mg$ for each link:

$$F_g = dm \cdot g = M\frac{d\ell}{L}g = \frac{Mg}{L}d\ell$$

Substituting $d\ell$ from Equation 1 above gives us:

$$F_g = \frac{Mg}{L}dx\sqrt{1 + (\frac{dy}{dx})^2}$$

Then from our balance of vertical forces, Equation 3 above, we get:

$$T_y(x + dx) - T_y(x) = \frac{Mg}{L}dx\sqrt{1 + (\frac{dy}{dx})^2}$$

We then divide both sides by dx:

$$\frac{T_y(x+dx) - T_y(x)}{dx} = \frac{Mg}{L}\sqrt{1 + (\frac{dy}{dx})^2}$$

Notice that $T_y(x + dx) - T_y(x)$ equals dT_y as used in derivatives from Calculus first principles – the change in vertical tension from one end of our short link to the other. So if we graphed tension T_y vs x then $\frac{T_y(x+dx) - T_y(x)}{dx}$ would be the slope $\frac{dT_y}{dx}$.

Our simplified equation for the balance of *vertical* forces then becomes:

$$\frac{dT_y}{dx} = \frac{Mg}{L}\sqrt{1 + (\frac{dy}{dx})^2} \qquad \text{(Equation 4)}$$

It is hard to see a path forward here, to find y(x), until we realize that the tension forces must be pulling in the same direction – or slope – as the chain link itself. Said another way, the ratio of the y and x components of the tension must equal the ratio of dy and dx:

$$\frac{T_y}{T_x} = \frac{dy}{dx}$$

It is tempting to make the left side above a derivative too, but remember that T_x is constant, so from Equation 2 we have:

$$\frac{T_y}{C} = \frac{dy}{dx} \quad \Rightarrow \quad T_y = C\frac{dy}{dx}$$

We do not know T_y, but we do know its derivative with respect to x, $\frac{dT_y}{dx}$, from Equation 4. So taking the derivative of both sides we have:

$$\frac{dT_y}{dx} = C\frac{d^2y}{dx^2}$$

And substituting from Equation 4:

$$\frac{Mg}{L}\sqrt{1 + (\frac{dy}{dx})^2} = C\frac{d^2y}{dx^2}$$

Solving for the 2nd derivative, we get:

$$\frac{d^2y}{dx^2} = \frac{Mg}{CL}\sqrt{1 + (\frac{dy}{dx})^2}$$

We are looking for y in terms of x to find the shape of the chain, so we are just two integrals away from our solution. To put this in the form of a first-order separable differential equation, we substitute $u(x) = \frac{dy}{dx}$ to get:

$$\frac{du}{dx} = \frac{Mg}{CL}\sqrt{1 + u^2}$$

Moving the u terms to one side, dx to the other, and integrating gives:

$$\int \frac{1}{\sqrt{1+u^2}}\, du = \frac{Mg}{CL}\int dx \qquad \text{(Equation 5)}$$

The term on the left looks familiar, so you may at first think that this can be solved by trigonometric substitution, but checking our table of standard derivatives shows that:

$$\frac{d}{dx}arccos(x) = \frac{1}{\sqrt{1-x^2}}$$ (Note: This does NOT help us!)

This derivative is close, but we're foiled by the minus sign in the denominator!

$\frac{d}{dx} arctan(x)$ is pretty close too, but is missing the square root. If we check a more complete table such as Wikibooks ("Tables of Derivatives"), however, we find our derivative:

$$\frac{d}{dx} arcsinh(x) = \frac{1}{\sqrt{1+x^2}}$$

Perfect! Well, except for that typo… the "h" in arcsin**h** must be just a typo, right?

Hyperbolic Functions – It's Not Just a Typo

It is time to admit that the shape of a hanging cable is not a parabola, and arcsin**h** is not just a typo. The "h" stands for "hyperbolic" and describes a whole new group of standard functions. Galileo knew this, and later in *The Two New Sciences* says that a hanging chain "closely approximates the parabola" (Galilei et al. 290). His earlier statement that it "assumes the form of a parabola" (Galilei et al. 149) is a simplification, similar to how we might say "Earth has a circular orbit" when we know full well it is elliptical. We will leave the integral unsolved for now, however, and explore these new hyperbolic functions. What are they, and how are they connected to parabolas and 'normal' trigonometric functions? These answers will eventually lead us back to our integral and our chain.

As we know, the normal trigonometric functions are closely related to the unit circle. But if we edit the equation for the unit circle, changing from addition to subtraction, then we have the equation for the unit hyperbola.

Unit Hyperbola equation: $x^2 - y^2 = 1$ ("Intuitive Guide").

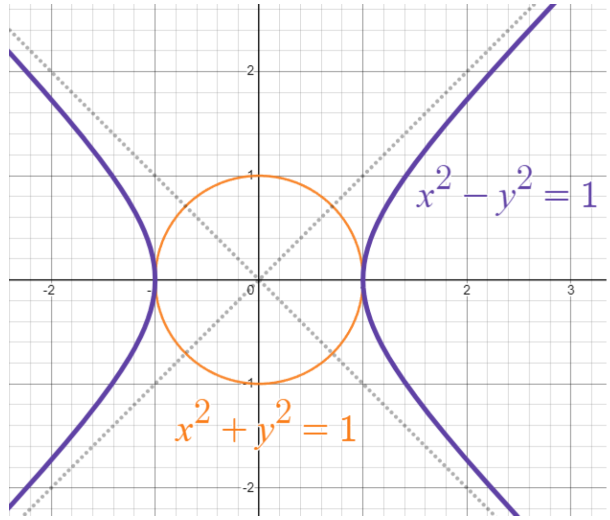

Figure 2: The Unit Hyperbola and the Unit Circle (Self-made on Desmos).

Graphs for both are shown in Figure 2; notice that the hyperbola approaches the lines $y = x$ and $y = -x$ asymptotically. The equation for a perhaps more familiar hyperbola is $y = \frac{1}{x}$; we get this if we rotate the graph around the origin until the asymptotes align with the x and y axes – or you can just tilt your head 45° to the right!

How are circles and hyperbolas related to the parabola, then? They are all conic sections, or slices of cones as shown in Figure 3. This explains why the parabola and hyperbola look so superficially similar; the parabola is the conic slice parallel to the ***edge*** of a cone. A hyperbola is any slice that

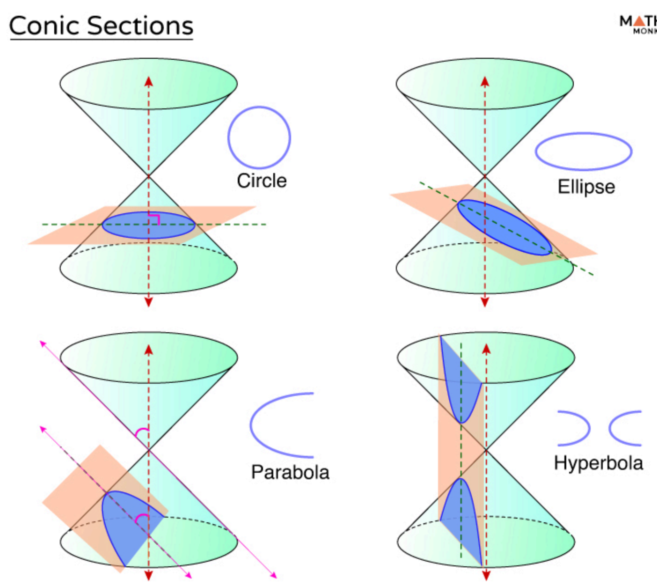

Figure 3: Conic sections: slices of a double-cone ("Conic Sections").

intersects both the upper and lower cones without passing through the vertex ("Difference between"). The *unit* hyperbola is the slice parallel to the ***axis*** of a cone, with a 90° vertex angle, at a distance of one unit from the axis ("12.2: The Hyperbola").

Let us return to our discovery of the sinh and cosh functions (Figure 4). Just as sine and cosine are the y and x values of a point on the circle, sinh and cosh are the y and x values of a point on the hyperbola. The equations – which shall be further explored later – are as follows ("Derivative of Hyperbolic"):

$$sinh(x) = \frac{e^{x} - e^{-x}}{2} \qquad \text{(Equation 6)}$$

$$cosh(x) = \frac{e^{x} + e^{-x}}{2} \qquad \text{(Equation 7)}$$

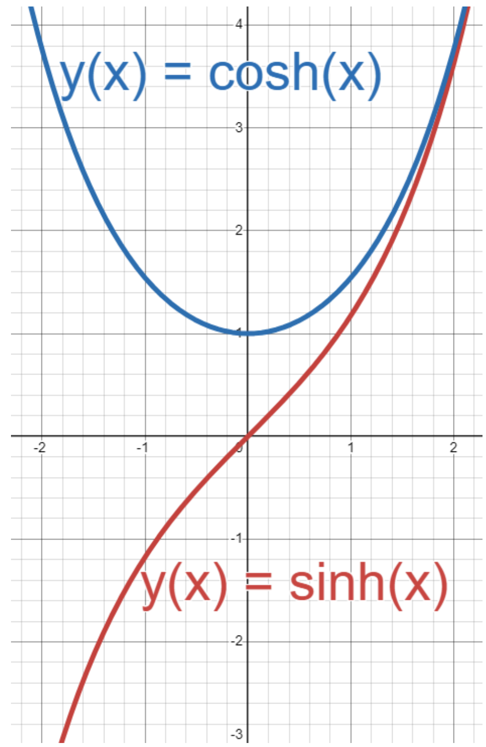

Figure 4: The sinh(x) and cosh(x) functions (Self-made on Desmos).

Note that although cosh(x) does look like it could ***be*** that slice of the cone shown in Figure 3, the unit hyperbola, it is not – it is the x-component of points along that hyperbola, just as the cosine is the x component of points along the unit circle. Any point (cosh(x) , sinh(x)) will fall on the unit hyperbola. And one last graphical note: cos(x) and cosh(x) are both even functions ($cosh(-x) = cosh(x)$) so are symmetrical over the y-axis, while both sin(x) and sinh(x) are odd ($sinh(-x) = -sinh(x)$) so are symmetrical about the origin. Let's now get back to the math of the functions themselves.

Why are sinh(x) and cosh(x) named so similarly to sin(x) and cos(x)? Their definitions in terms of the exponential functions (Equations 6 and 7) might not give us a clue until we remember Euler's Formula ("Intuitive Guide"):

$$e^{i\theta} = cos(\theta) + isin(\theta)$$

And so, because cosine is even and sine is odd:

$$e^{i(-\theta)} = cos(-\ \theta) + isin(-\ \theta) = cos(\theta) - isin(\theta)$$

Adding these 2 variations of Euler's formula gives us:

$$e^{i\theta} + e^{i(-\theta)} = 2cos(\theta) + isin(\theta) - isin(\theta) = 2cos(\theta)$$

$$cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Similarly, subtracting the two gives us:

$$e^{i\theta} - e^{i(-\theta)} = cos(\theta) - cos(\theta) + 2isin(\theta) = 2isin(\theta)$$

$$sin(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2i}$$

These are promising, but the imaginary unit isn't found in the definitions of cosh(x) and sinh(x), so let's fix that by non-intuitively adding an i to get rid of an i, by setting

$\theta = ix$:

$$sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i} \qquad cos(ix) = \frac{e^{i(ix)} + e^{i(-ix)}}{2} = \frac{e^{-x} + e^{x}}{2}$$

And then rearranging a bit so the right sides of these equal the right sides of the hyperbolic functions above gives us:

$$-\ isin(ix) = \frac{e^{x} - e^{-x}}{2} = sinh(x) \quad cos(ix) = \frac{e^{x} + e^{-x}}{2} = cosh(x)$$

So sinh(x) and cosh(x) are revealed to be very similar to sin(x) and cos(x) if we dip our toes into the imaginary plane. They can be rewritten using the notation in orange above, but they are common enough in physics and mathematics that they have earned their own names. And it is important to note that despite i appearing 3 times in the formulas above, both the inputs and outputs are real.

Now let us consider the *derivatives* of our hyperbolic functions – they are simple to calculate now that we know them in terms of the exponential function, and we will eventually need them in order to analyze our almost-forgotten chain.

$$\frac{d}{dx} sinh(x) = \frac{d}{dx}\left(\frac{e^{x} - e^{-x}}{2}\right) = \frac{1}{2}\left(\frac{d}{dx}e^{x} - \frac{d}{dx}e^{-x}\right) = \frac{1}{2}([e^{x} - (-\ e^{-x})]$$

$$= \frac{e^{x} + e^{-x}}{2} = cosh(x)$$

So like cos(x) is the derivative of sin(x), cosh(x) is the derivative of sinh(x).

$$\frac{d}{dx} cosh(x) = \frac{d}{dx}\left(\frac{e^{x} + e^{-x}}{2}\right) = \frac{1}{2}\left(\frac{d}{dx}e^{x} + \frac{d}{dx}e^{-x}\right) = \frac{1}{2}([e^{x} + (-\ e^{-x})]$$

$$= \frac{e^{x} - e^{-x}}{2} = sinh(x)$$

So we see that in both cases, all that changes when differentiating sinh(x) or cosh(x) is the sign (not sine!) of the second term, e^{-x}, and the sinh(x) and cosh(x) functions alternate from one to the other on repeated differentiation. It is interesting to note that there is therefore no sign change as when differentiating cos(x) to -sin(x).

These derivatives will be helpful, but they don't yet help with our catenary integral because – as our table of derivatives indicated – we need the derivative of ***arc***sinh(x), not just sinh(x). We could just trust the table, now that we know that arcsinh(x) isn't a typo, but a proof will provide a deeper understanding.

Let us start by determining exactly what arcsinh(x) is by starting with our original function:

$$sinh(x) = y = \frac{e^{x} - e^{-x}}{2}$$

To find the inverse of the function, we swap the variables x and y and then solve for y:

$$x = \frac{e^{y} - e^{-y}}{2}$$

$$2x = e^{y} - e^{-y}$$

Multiply both sides by e^{y} to cancel with the e^{-y}:

$$2xe^{y} = (e^{y} - e^{-y})e^{y} = e^{2y} - e^{y-y} = e^{2y} - 1$$

We then move everything to one side, and adjust a bit:

$$e^{2y} - 2xe^{y} - 1 = (e^{y})^{2} - 2x(e^{y}) - 1 = 0$$

We can now see that we have a quadratic equation with $a = 1,\ b =- 2x\ and\ c =- 1.$

Using the quadratic formula and substituting our $a,\ b,$ and c values gives us:

$$e^{y} = \frac{-b \pm \sqrt{b^{2} - 4ac}}{2a} = \frac{-(-2x) \pm \sqrt{(-2x)^{2} - 4(1)(-1)}}{2(1)} = \frac{2x \pm \sqrt{4x^{2} + 4}}{2} = \frac{2x \pm \sqrt{4(x^{2}+1)}}{2}$$

$$e^{y} = \frac{2x \pm 2\sqrt{x^{2}+1}}{2} = x \pm \sqrt{x^{2} + 1}$$

Let us consider the $\pm$ sign – perhaps we can discount one of the possibilities? We know that $\sqrt{x^{2}} = x$ so $\sqrt{x^{2} + 1} > x$. Therefore $(x - \sqrt{x^{2} + 1})$ must be negative, but $e^{y} > 0$ for all y. So we know that $e^{y} \neq x - \sqrt{x^{2} + 1}$ and our equation above simplifies to:

$$e^{y} = x + \sqrt{x^{2} + 1}$$

We can then take the natural log of both sides:

$$ln(e^{y}) = ln(x + \sqrt{x^{2} + 1})$$

And this gives us our result for arcsinh(x):

$$y = arcsinh(x) = ln(x + \sqrt{x^{2} + 1})$$

Let's continue, using the chain rule twice to find the derivative!

$$\frac{d}{dx} arcsinh(x) = \frac{d}{dx} ln(x + \sqrt{x^{2} + 1})$$

$$= \frac{1}{x + \sqrt{x^{2}+1}} \frac{d}{dx}\left(x + (x^{2} + 1)^{\frac{1}{2}}\right) = \frac{1}{x + \sqrt{x^{2}+1}}\left(1 + \frac{1}{2\sqrt{x^{2}+1}} \frac{d}{dx}(x^{2} + 1)\right)$$

$$= \frac{1}{x + \sqrt{x^{2}+1}}\left(1 + \frac{2x}{2\sqrt{x^{2}+1}}\right) = \frac{1}{x + \sqrt{x^{2}+1}}\left(\frac{\sqrt{x^{2}+1}}{\sqrt{x^{2}+1}} + \frac{x}{\sqrt{x^{2}+1}}\right)$$

$$= \frac{1}{x + \sqrt{x^{2}+1}}\left(\frac{\sqrt{x^{2}+1} + x}{\sqrt{x^{2}+1}}\right) = \frac{1}{\sqrt{x^{2}+1}}$$

I love it when huge parts of complicated equations simply cancel out! We have now explored a new group of functions: hyperbolic functions. We have also proven that the table entry we found for our integral is indeed true, and can now use it with confidence.

$$\frac{d}{dx} arcsinh(x) = \frac{1}{\sqrt{x^2+1}}$$

Return to the Hanging Chain

We had left our chain hanging. We were stymied by our lack of knowledge of the hyperbolic functions, so our integrand probably thought it had escaped being chopped into infinite little pieces of width dx and summed, but we're back. So where were we?

Equation 5: $\int \frac{1}{\sqrt{1+u^2}} du = \frac{Mg}{CL} \int dx$ Where $u(x) = \frac{dy}{dx}$

The work we put in above now makes this easy.

$arcsinh(u) = \frac{Mg}{CL}x + A$ where A is a constant of integration.

Taking the hyperbolic sine of both sides cancels out the $arcsinh(u)$ leaving us:

$$u(x) = sinh\left(\frac{Mg}{CL}x + A\right)$$

Substituting back $u(x) = \frac{dy}{dx}$ gives us:

$$u(x) = \frac{dy}{dx} = sinh\left(\frac{Mg}{CL}x + A\right) \qquad \text{(Equation 8)}$$

Another integration – this time not requiring a 7-page detour – gives us:

$$\int dy = \int sinh\left(\frac{Mg}{CL}x \ + A\right)dx$$

$$y = \frac{CL}{Mg}cosh\left(\frac{Mg}{CL}x \ + A\right) + B$$ where B is another constant of integration.

$A,\ B,$ and $C \in \mathbb{R}$. We now have our general solution for the shape of a hanging chain!

Figure 5: The Geierlay suspension bridge in Western Germany ("Geierlay").

The Geierlay Suspension Bridge

We do have some loose ends: our constants A, B, and C. A bridge with loose ends sounds dangerous, so let's tie them down. They can be resolved only for an actual catenary, so we will evaluate them for the Geierlay suspension bridge (Figure 5). We can use the real-life parameters of this bridge to determine our initial conditions:

$$Start\ Point\text{:}\ (0, 0)\ \ End\ Point\text{:}\ (360, 0)\ \ giving\ us\ a\ Span\ of\ 360\ meters$$

$$M = 57,000\,kg \quad L = 374\,m \quad g = 9.81\,ms^{-2}$$ ("Geierlay Suspension")

Using this information and our general solution y(x), we can find three equations with three unknowns, A, B, and C.

First with the point (0,0): $0 = \frac{CL}{Mg}cosh(A) + B$

Second, with the point (360,0): $0 = \frac{CL}{Mg}cosh(\frac{360 \cdot Mg}{CL} + A) + B$

We now must consider the total arc length of the bridge, L. We'd introduced this as a constant, but now we must determine an equation which we can use as our third initial condition equation. To calculate our total length L we must integrate all dl, the infinitely small lengths of each chain link, and from Equation 1 we find:

$$L = \int dl = \int \sqrt{1 + (\frac{dy}{dx})^2}dx$$

From Equation 8 we get:

$$L = \int_0^{360} \sqrt{1 + sinh^2\left(\frac{Mg}{CL}x + A\right)}dx$$

Using the hyperbolic identity $cosh^2(x) - sinh^2(x) = 1$ (Branson) we get:

$1 + sinh^2\left(\frac{Mg}{CL}x + A\right) = cosh^2\left(\frac{Mg}{CL}x + A\right)$, so substituting:

$$L = \int_0^{360} \sqrt{cosh^2\left(\frac{Mg}{CL}x + A\right)}dx = \int_0^{360} cosh\left(\frac{Mg}{CL}x + A\right)dx$$

Our third condition then simplifies to:

$$L = \left[\frac{CL}{Mg} sinh(\frac{Mg}{CL}x \ + \ A)\right]_0^{360} = \frac{CL}{Mg}\left[sinh(\frac{Mg}{CL}360 \ + \ A) \ - \ sinh(A)\right]$$

If these equations were polynomials, then this system of equations would be simple to solve, but they are not. Most analyses of the Catenary problem end here, stating that these constants are solved “numerically” ("Worked Example"). For modern engineers, this means they use software tools to calculate these values for given initial conditions. This software is based on “the Runge-Kutta methods”, including Euler’s method (Barkley 2-3), which iterates repeatedly, and uses the slope to determine the direction to move for the next iteration until a solution is found.

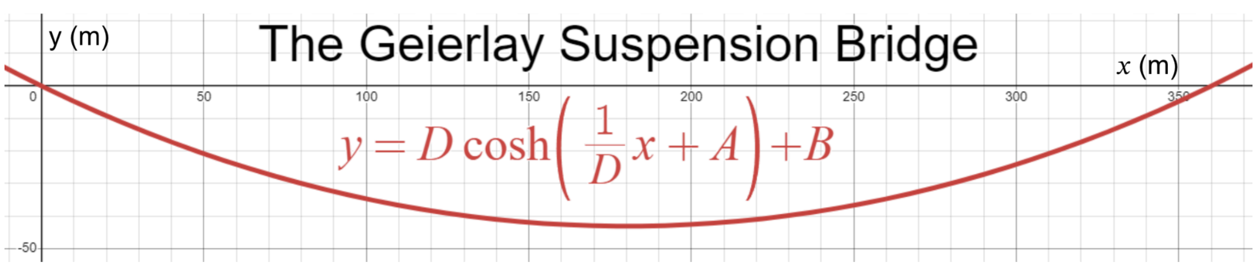

Figure 6: The graph and equation for the Geierlay bridge (Self-made on Desmos).

I determined the following values to achieve the graph shown in Figure 6:

$A \approx -\ 0.47 \qquad B \approx -\ 426.1\ m \qquad D \approx 383\ m$

C then is: $C \ = \frac{DMg}{L} \approx \frac{(383)(57{,}000)(9.81)}{374} \approx \ 572{,}626\ N$

So now that we have calculated our constants for the Geierlay bridge, we can offer our final equation, graphed in Figure 6:

$$y \ = \ 383cosh\left(\frac{x}{383} \ - \ 0.47\right) - \ 426.1$$

This equation can calculate where each step you take on the Geierlay bridge is located! But an even more practical result is that these coefficients can help determine the very materials the bridge must be built with. Our constant $C = 572,626\ N$ is the horizontal tension at each point on the bridge and when multiplied by $\frac{dy}{dx}$ at a point gives us the vertical tension (see the development of Equation 2 on Page 5). The anchor posts and cables must be able to withstand these forces in addition to external forces like wind and pedestrians. This information is critical to building a safe and reliable bridge.

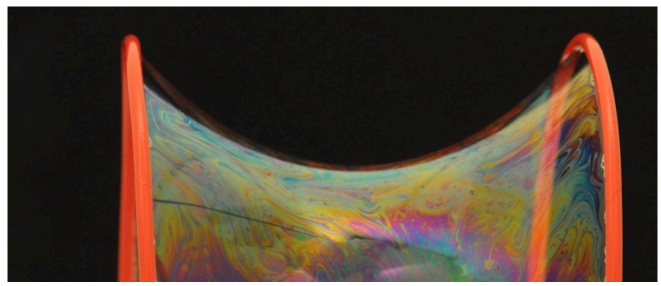

Figure 7: Soap film coating two rings forms a revolution of the cosh(x) function: a catenoid (Minimal Surface).

<u>**Conclusion – It's Not Just a Chain**</u>

The hanging chain problem has introduced us to hyperbolic functions, which we will no doubt encounter again as we proceed with mathematics and physics. A hanging chain minimizes potential energy – if a disturbance moves the chain away from this shape, then it will return when that disturbance is removed ("The Math"). The cosh(x) function will also be encountered when we investigate minimal surfaces – ***surfaces*** that minimize potential energy. Soap film forms natural minimal surfaces: for example, if two rings are dipped in soap and separated (see Figure 7) then they form a catenoid ("The

Math"). This is a cosh(x) function revolved around a horizontal axis. It forms this shape in order to minimize the potential energy of the surface.

It now seems obvious that this shape is not a parabola, but based on cosh(x) which is related to the hyperbola. Both the hyperbola and the parabola are conic sections, but if we had accepted the obvious but wrong answer without proof, then we would have missed out on this fascinating discovery. Mathematics, much like the Geierlay Bridge, depends on what lies beneath the surface: the cables and anchors support the bridge, while proofs support our conclusions.

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