# The Reality Of Imaginary Numbers

To what extent can imaginary numbers be considered real due to their usefulness in solving real-world problems?

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# Introduction: Real and Imaginary Numbers

Mathematics is used to solve real-world problems, and is considered most valid when it can accurately do so. Real-world problems are translated into – or modeled by – mathematical equations, solved, and then the results of the model are interpreted as real-world results. One trivial example of this is “Sally has 3 apples. She picks 4 more. How many apples does Sally have now?” This can be modeled as 3 + 4, then solved to equal 7, and interpreted as “Sally has 7 apples”. From this simple beginning math has been essential to understanding the deepest mysteries of the universe – from the Earth’s core out to the most distant stars, from subatomic particles to supermassive black holes, and from the number of apples Sally has picked to the most complex economies of the world.

Many modern mathematicians would agree that imaginary numbers are poorly named, but they were named in contrast to real numbers. But are real numbers any more real? Are even the simplest numbers, counting numbers $1, 2, 3$… , “real”? You can’t eat them, so they are not real in the normal sense of the word, but they are an exact representation of the real world. It must be accepted, then, that the reality of numbers is judged not by their ability to make apple pie, but by the accuracy with which they model reality: their usefulness. From counting numbers, the set $\mathbb{R}$ continued to grow. Zero and then fractions started to fill in the real number line and clearly model reality. Irrational numbers were the first to be met with serious controversy (Kathotia), but $\sqrt{2}$ perfectly represents the diagonal of a unit square: rounding it would model reality less accurately, and is therefore less real. Negative numbers met with more controversy, but extended the real number line to infinity in a second direction and allowed the modeling of new realities. When adequately defined, for example as the distance in the direction opposite that defined as positive, then negative numbers are an exact model of reality. So do not be fooled that real numbers are real; they lie on a spectrum from more obvious to less, but none of them can be picked up, handled, or eaten. Real numbers are simply the numbers that best model the most obvious real-world problems. As mathematics took on new challenges, so did the types of numbers that were required to solve them, but that evolution did not end when the real number line was filled in.

The history of imaginary numbers parallels that of some types of real numbers: controversy until their usefulness was proven, and then acceptance. They are no less real, however, than real numbers; you can't eat $3i$ apples any more than you can eat 0, -3, or *exactly* $\sqrt{2}$ apples. So imaginary numbers are but one more addition to established number space, and inevitably lead to complex numbers.

There is no doubt that complex numbers often make calculations more complicated, but that is not why they are called 'complex'. They are complex because they are multi-part, like a "building complex" is a system of interconnected buildings. They could as easily be called duplex numbers – two-part – and with less confusion. These two-part numbers allow mathematicians to model quantities that are too complicated for a single real number. And the fact that $\sqrt{-1}$, on repeated multiplication, rotates around the complex plane in an orderly fashion already hints that complex numbers may accurately represent circular phenomena.

Another idea must be briefly investigated: mathematical operations. When new types of numbers were adopted, new operations were adopted or adapted. When zero was accepted, dividing by it was defined as undefined. Many operations were modified to work with fractions and negative numbers. One particularly illuminating example is the unsigned 'size' of a number. Before negative numbers, a number's magnitude exactly equaled the number. When negative numbers were accepted, the concept of 'absolute value' was introduced to define the distance a number is from zero. With complex numbers the absolute value is still the distance from zero, but the formula is now more complicated:

Absolute value or modulus of a complex number: $|z| = \sqrt{x^2 + y^2}$

$$for\, z = x + yi \;\; where \;\; x, y \in \mathbb{R}$$

Complex numbers do indeed make some operations more complex, but it is satisfying that this new equation for absolute value does simplify to the previously-established rule of dropping the negative sign if a complex number falls on the real number line:

$$when\, y = 0,\; then: \;\; |z| = \sqrt{x^2 + 0^2} = \sqrt{x^2} = |x|$$

Mathematicians have developed new types of numbers throughout history. Each new type was met with resistance and skepticism until they proved their worth in solving real-world problems. A line was drawn, however, between real and imaginary numbers that stigmatizes the latter as somehow less 'real' than other numbers. So it must be asked: To what extent can imaginary numbers be considered real due to their usefulness in solving real-world problems? This question will be analyzed in detail using reliable secondary sources from diverse fields, and the usefulness and reality of imaginary numbers will be proven.

## Euler’s Formula

To prove the usefulness of imaginary numbers, one must first investigate their fundamental, rotary nature articulated in Euler’s formula. Born in Basel Switzerland, Leonhard Euler was one of the greatest mathematicians of all time. Despite struggling with poor eyesight throughout his life, and eventually becoming almost blind (Nagrath), he provided one of the greatest insights into complex numbers, Euler’s Formula:

$$e^{i\theta} = cos\theta + i\,sin\theta \quad (\theta \in \mathbb{R}, \, and\ i\ is\ the\ imaginary\ unit,\ \sqrt{-1})$$

The first objection one might make is “What does the exponential function have to do with sinusoidal functions? They look nothing alike!”. On closer analysis, one might then ask “How do you multiply $e$, Euler’s Number, by itself an imaginary number of times?” These questions can only be answered by completely reevaluating the typical understanding of $e^{x}$.

Just as the concept of absolute value had to be redefined for complex numbers, the exponential function must be redefined as well. It will henceforth be written as $exp(z)\ (where\ z \in \mathbb{C})$ rather than $e^{x}\ (where\ \ x \in \mathbb{R})$ to distinguish between the new definition that works with complex numbers and the old definition that does not.

There have been many definitions of the exponential function, but the one that is most useful for understanding Euler's formula is the Maclaurin Series for $exp(z)$, which is defined as follows:

$$exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \frac{z^0}{0!} + \frac{z^1}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + ...$$

$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + ... \qquad where\ \ z \in \mathbb{C}$$

A Maclaurin series is a special case of a Taylor Series, which is an infinite sum that is equivalent to the function.

A rigorous proof that this formula equates to $e^x$ for real numbers is beyond the scope of this essay, but it is demonstrated graphically for real inputs $x$ in Figure 1.

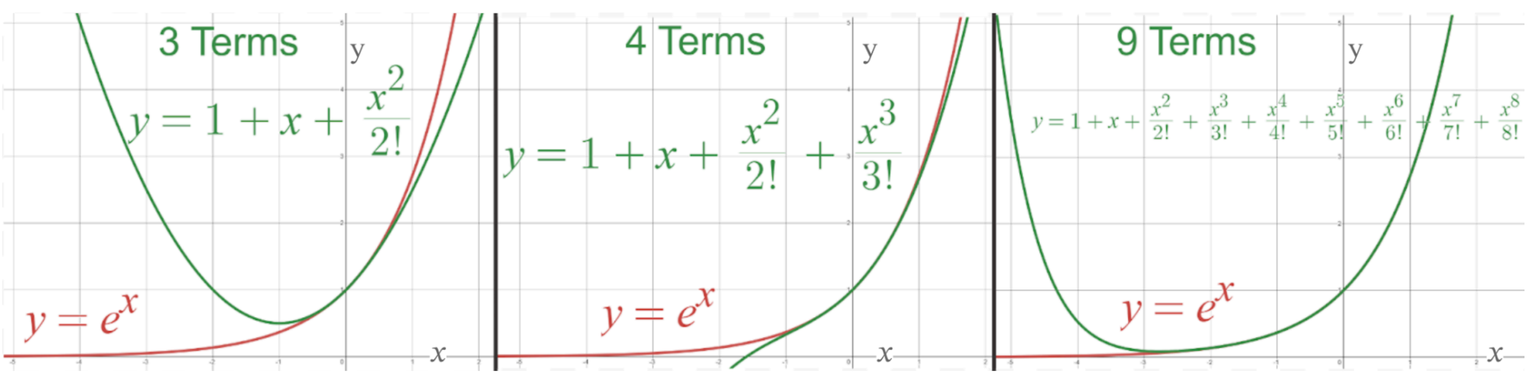


Figure 1: Maclaurin series for $exp(x)\ when\ x \in \mathbb{R}$, graphed using 3, 4 and 9 terms. Created on Desmos.

By as few as 9 terms the familiar shape of $y = e^x$ is emerging, though many more terms are required for it to converge for large negative values. It is important to note that only real values are being input here to show that the new formula works the same as the old for real numbers. The magic happens, however, when plotting $exp(i\theta)$ for purely imaginary inputs ($when\ \theta \in \mathbb{R}$) which the new definition allows: as the inputs move up the imaginary axis, the result is repeated circuits around the unit circle on the complex plane, and θ is the angle in radians.

So $exp(x) \ = e^{x}$ for real numbers, but does it satisfy another very important characteristic of the exponential function: is its derivative equal to the function itself? The derivative of the first term of the infinite sum, 1, is zero. The derivative of the other terms can be calculated as follows:

$$\frac{d}{dz}exp(z) \ = \frac{d}{dz}\sum_{n=0}^{\infty}\frac{z^{n}}{n!} = \sum_{n=0}^{\infty}\frac{d}{dz}\frac{z^{n}}{n!} = \sum_{n=0}^{\infty} n\frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty}\frac{z^{n-1}}{(n-1)!}$$

This uses the Linearity Property of derivatives and the definition of a factorial: the n in the numerator cancels with the highest term of the factorial, leaving $(n\ -\ 1)!$. Then the derivative of each term is the previous term: $z$ becomes 1, replacing the original 1, $\frac{z^{2}}{2!}$ becomes $z$, $\frac{z^{17}}{17!}$ becomes $\frac{z^{16}}{16!}$ and so on. There is one less term, but since ∞ means the series can continue as long as necessary to converge, it does not change the result. Thus the new exponential function is still a satisfactory replacement for $e^{x}$.

Before continuing the analysis of Euler’s Formula, the Maclaurin series for sine and cosine are required:

$$cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \frac{z^{12}}{12!} - ...$$

$$sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \frac{z^{13}}{13!} - ...$$

It is fascinating to graph these for real values, adding each successive term, and see the infinitely repeating sinusoids being constructed piecemeal by polynomials. The reader is encouraged to do so more thoroughly using an online graphing calculator. However, only 3 instances for cos(x) are shown here (see Figures 2, 3 and 4).

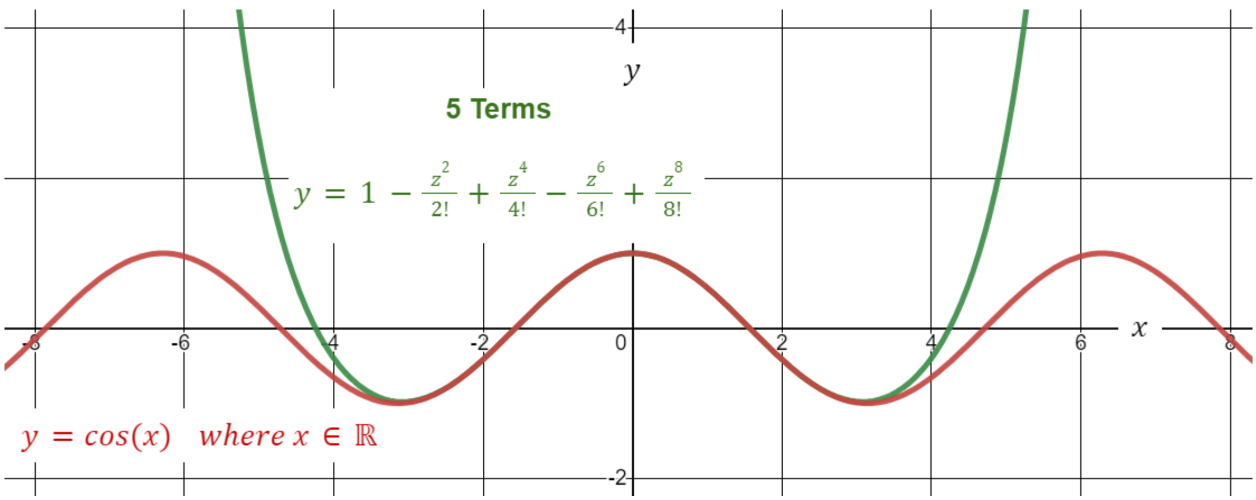


Figure 2: Maclaurin series for $cos(x)$ using 5 terms. Created on Desmos.

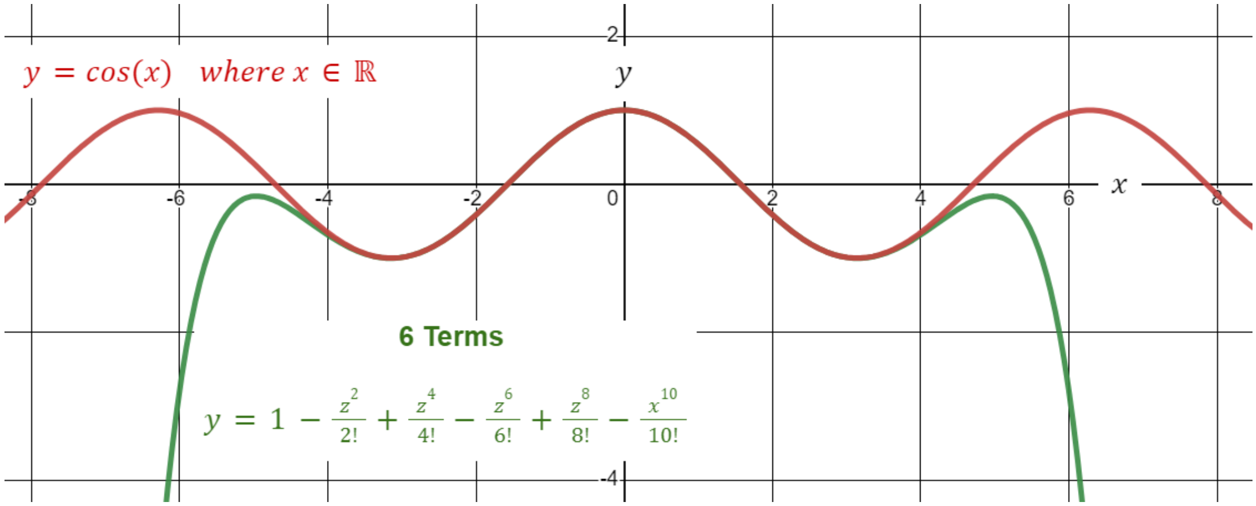


Figure 3: Maclaurin series for $cos(x)$ using 6 terms. Created on Desmos.

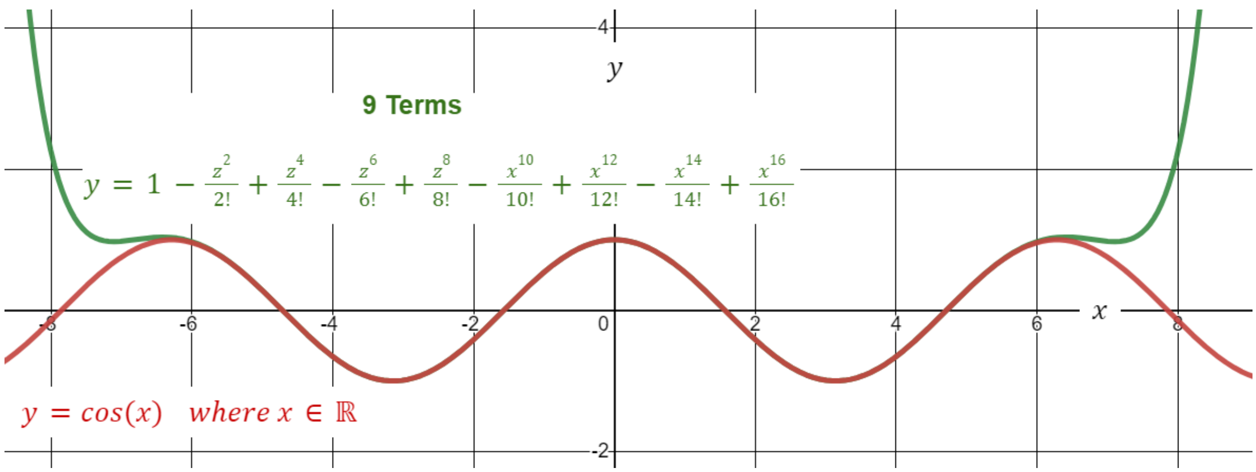


Figure 4: Maclaurin series for $cos(x)$ using 9 terms. Created on Desmos.

All the tools required to prove Euler's Formula are now at hand.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Substituting the Maclaurin series for $cos\theta$ and $sin\theta$, and then distributing $i$:

$$RHS = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots + i[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots ]$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + ... + i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} - \frac{i\theta^7}{7!} + ... ]$$

The Maclaurin series is substituted on the left hand side, and $i^n$ resolved to 1, $i$, -1 or -$i$:

$$LHS = exp(i\theta) = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + ...$$

$$= 1 + i\theta + \frac{(-1)\theta^2}{2!} + \frac{(-i)\theta^3}{3!} + \frac{(1)\theta^4}{4!} + \frac{i\theta^5}{5!} + \frac{(-1)\theta^6}{6!} + \frac{(-i)\theta^7}{7!} ...$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + ...$$

$LHS = RHS$ (Thibault)

This is one proof of Euler’s Formula out of many, and it was chosen because it can be deconstructed into individual terms and analyzed. To do exactly that, here is $exp(i\pi)$:

$$exp(i\pi) \approx 1 + i\pi - 3.935 - 2.026i + 0.124 + 0.524i - 1.211 - 0.075i ...$$

This could of course be written more simply by summing all the real and imaginary parts, but in Figure 5 the sum is shown as each term is added individually (Sanderson, “What is” 33:00 - 39:18).

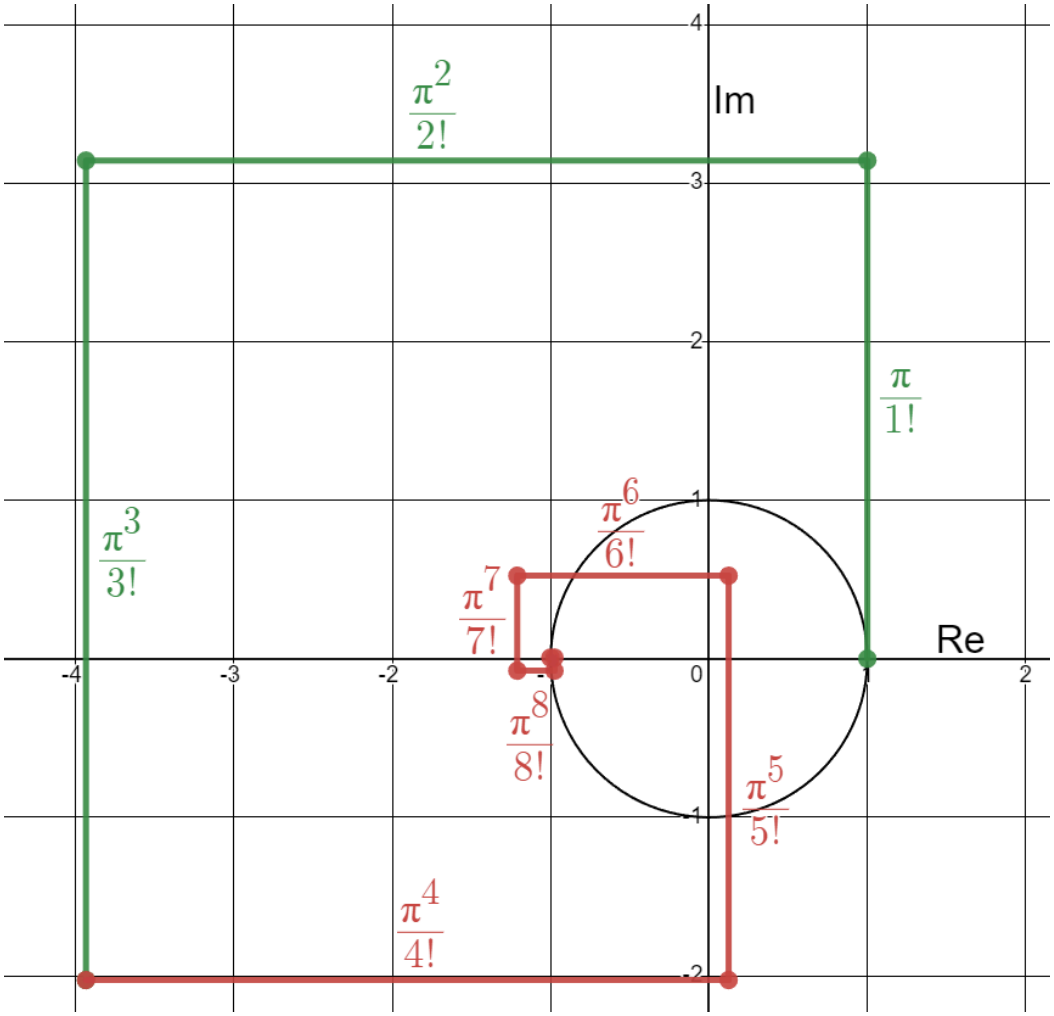


Figure 5: $exp(i\pi)$ showing a line segment (lengths labeled) for each term of the Maclaurin series. Created on Desmos.

It starts at $(1,\ 0i)$ on the complex plane's unit circle, and the terms get bigger as $\pi^n$ in the numerator grows faster, at first, than the $n!$ in the denominator. **These increasing values – when each term is bigger than the previous term – are shown as green line segments.** But once $n$ becomes greater than $\theta$ **the terms start to decrease in size and these segments are shown in red**. They converge to a different point on the unit circle – in this case to (-1, 0). They spiral out (**green**) and then back in again (**red**), and always converge onto a point on the unit circle – quite a beautiful and fascinating result. Note that the vertical line segments are the imaginary components that equal the Maclaurin series for $sin(\theta)$, and the horizontal segments correspond to $cos(\theta)$. This is Euler's Formula, $e^{i\theta} = cos\theta + isin\theta$, deconstructed and laid bare!

This particular instance, when $\theta = \pi$, is known as Euler's *Identity* and is usually written as $e^{\pi i} = -1$. It is famous because it has many important mathematical constants, $e$, $\pi$, $i$, and -1 all in a single equation. It has been demonstrated that it actually has little to do with the constant $e$, but knowing that $e$ is replaced by this gloriously spiraling Maclaurin series only adds to the luster of this identity.

One more graph of this fascinating result must be examined before moving on, and a bit more can be learned from it. In this case $exp(i\theta)$ is graphed when $\theta = 2\pi$. As shown in Figure 6, the values get much larger before spiraling back in – the unit circle is barely visible at the center – but they converge right back to where they started, at $(1,\ 0i)$. No matter how far $exp(i\theta)$ spirals out, it always spirals back in to the unit circle.

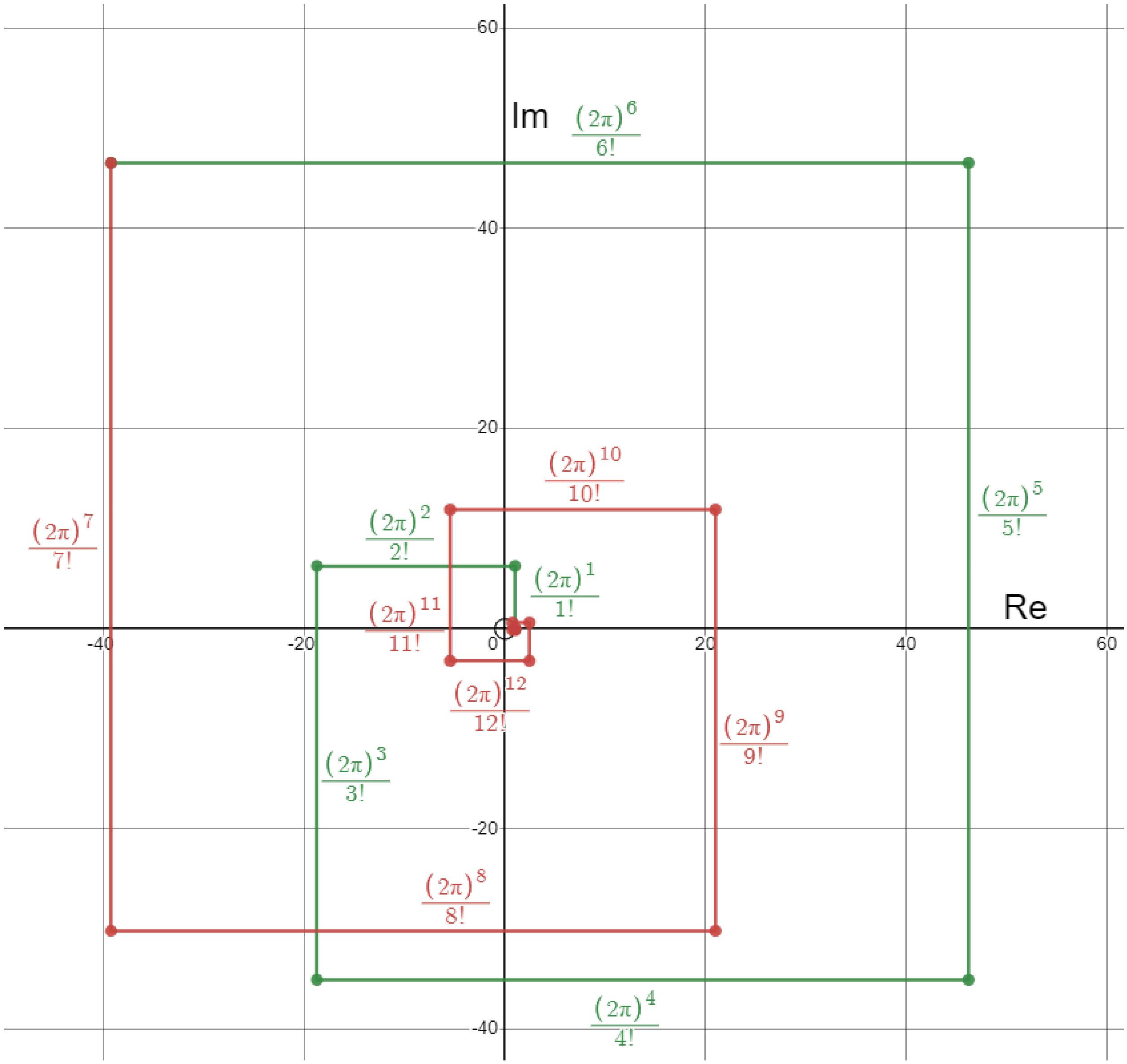


Figure 6: $exp(i \cdot 2\pi)$ showing a line segment (lengths labeled) for each term of the Maclaurin series. Created on Desmos.

## Steinmetz and Phasor Algebra

An argument has been made that complex numbers are just as real as real numbers if they can accurately model real-world problems. Charles Proteus Steinmetz found exactly such a real-world problem. Steinmetz was only 4 feet in height physically, but he was a titan of engineering. He worked for General Electric and was instrumental in improving power distribution methods using alternating current (AC) (The Editors of Encyclopaedia Britannica). One of his most brilliant insights was that AC circuits could be modeled using complex-valued quantities.

A capacitor is an electrical component that stores energy by accumulating electrical charge on two closely-spaced but insulated surfaces – it opposes a change in voltage, and therefore delays the phase of an AC ***voltage*** applied to it by 90° (Kuphaldt Chapter 3). An inductor is an electrical component that stores energy in a magnetic field using a coil of wire – it opposes changes in current, and therefore delays the phase of ***current*** by 90° (Kuphaldt Chapter 4). The resistor is the third fundamental component of electrical circuits and does not change the phase of an input voltage.

In Steinmetz's method, inductance (L) is treated as a positive imaginary number, capacitance (C) as a negative imaginary number, and resistance (R) as a positive real number (Kuphaldt Chapter 5). All these values are collectively called impedances (Z) and must be added and multiplied as complex numbers (MIT OpenCourseWare). Ohm's Law is then rewritten as follows: $V = IZ$, where V is voltage and I is current (Elinoff and Keim Chapter 3). $Z_C$ and $Z_L$ depend on ω, the angular frequency of the AC voltage, as shown in Figure 7. When in exponential form, θ is simply the phase shift of a sinusoidal value in the circuit relative to the reference, the input voltage, which is defined as 0°.

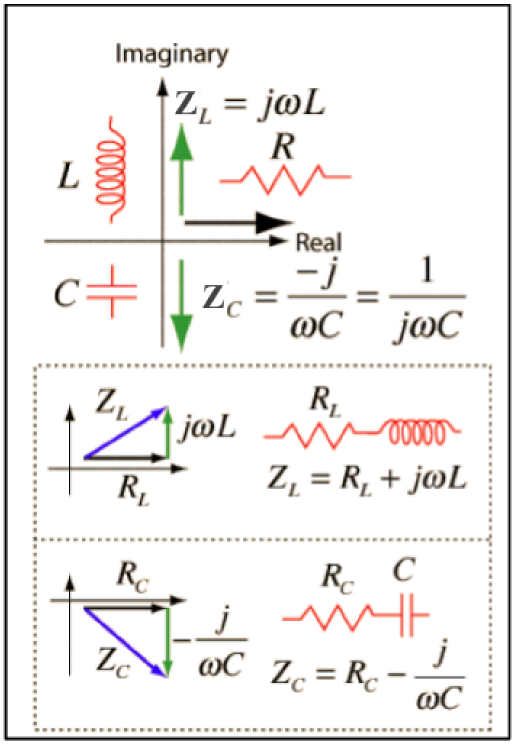


Figure 7: Impedance, Z, based on angular frequency ω. Image from Georgia State University Hyperphysics (Nave).

AC circuit analysis had previously been performed using calculus and trigonometric identities, which is just as complicated as it sounds, so Steinmetz's method greatly simplified calculations. His method is so powerful, in fact, that it has almost completely replaced the old method, which is rarely taught in universities anymore (Araújo and Tonidandel).

The power of Steinmetz's model is demonstrated here with an example, though some understanding of circuit analysis is assumed. Consider the circuit in Figure 8.

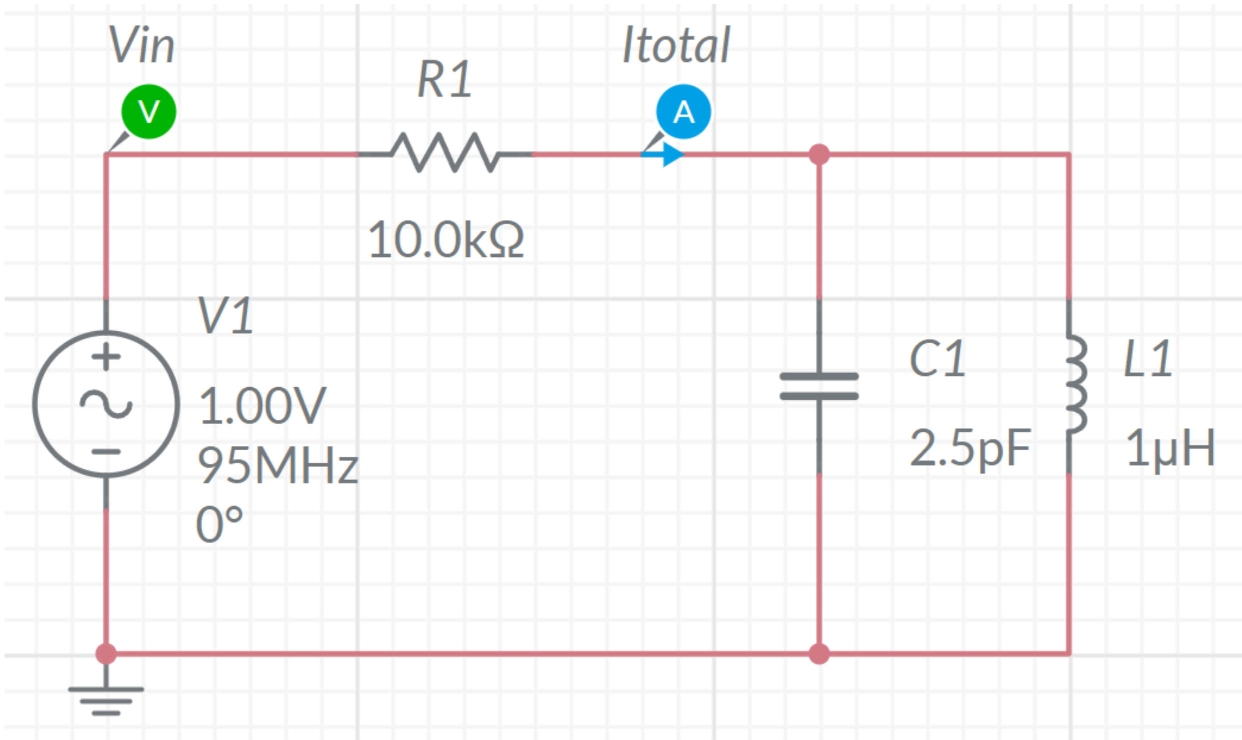


Figure 8: A simple AC circuit. L1 is an inductor, C1 a capacitor, V1 the battery, R1 a resistor. Created using Multisim.

This circuit will be analyzed using Steinmetz's method (MIT OpenCourseWare 18), also known as phasor algebra, but note that $j$ is used as the imaginary unit as is common in electrical analysis to avoid confusion with current, which is also $i$.

Impedance of the Capacitor and Inductor are:

$$Z_c = \frac{-j}{2\pi f C} = \frac{-j}{2\pi(95\times10^{6})(2.5\times10^{-12})} \approx -670.1j = 670.1e^{j(\frac{-\pi}{2})}\Omega$$

$$Z_L = j(2\pi f L) = j(2\pi)(95 \times 10^{6})(1 \times 10^{-6}) \approx 596.9j = 596.9e^{j(\frac{\pi}{2})}\Omega$$

The equivalent impedance of these parallel components is then:

$$Z_{CL} = \frac{1}{\frac{1}{Z_C}+\frac{1}{Z_L}} = \frac{1}{1.49\times10^{-3}e^{j(\frac{\pi}{2})}+1.68\times10^{-3}e^{j(\frac{-\pi}{2})}} = \frac{1}{1.83\times10^{-4}e^{j(-\frac{\pi}{2})}} \approx 5,463e^{j(\frac{\pi}{2})}\Omega$$

Total Impedance is then:

$$Z_T = R_1 + Z_{CL} = 10,000 + 5,463i \approx 1.139 \times 10^4 e^{j(.50)}\Omega$$

Current is then: $I = \frac{V}{Z_T} = \frac{1.00}{1.139\times10^4 e^{j(.50)}} \approx 0.8776e^{j(-.50)}\ mA$

This means that the current has an amplitude of about 0.88 mA and lags 28.7° (.50 radians ≈ 28.7°) behind the input voltage. No calculus was needed to obtain this result, so it is much more efficient than the calculus-heavy method used prior to Steinmetz. Only addition, multiplication, and division of complex numbers is required.

So the quantities of a real-world circuit were transformed into complex numbers, solved, and the result was interpreted to find a real-world answer. This is exactly how Sally's apple problem was solved. Is it a coincidence that complex numbers can so perfectly model AC circuits? To answer that, perhaps one must first answer whether it's a coincidence that $3 + 4 = 7$ perfectly models Sally's apples. Complex numbers lend themselves to representing complex values that are interrelated in a circular or rotary way, and phase shift in AC circuits is exactly such a quantity.

# The Fourier Transform

The Fourier Series is a revolutionary idea that enables mathematicians and scientists to describe any function as a sum of sinusoidal waves of different frequencies. It is named for Joseph Fourier, who participated in the French Revolution, and even traveled to Egypt for a time as Napoleon's scientific advisor (Struik). Though he died before the Fourier Series and the related Fourier Transform were fully developed by later mathematicians, these revolutionary ideas still bear his name.

The Fourier Series is the series of sinusoidal waves that add to a function, and all functions can be represented as such series (Osgood, "Stanford EE261"). This is again demonstrated graphically in lieu of a formal proof. Consider a sawtooth wave (Figure 9, shown in black); the equation for the Fourier Series for this wave is:

$$\sum_{N=1}^{\infty} \frac{(-1)^{N+1}}{N} sin(Nt) = sin(t) - \frac{1}{2}sin(2t) + \frac{1}{3}sin(3t) - \frac{1}{4}sin(4t)...$$

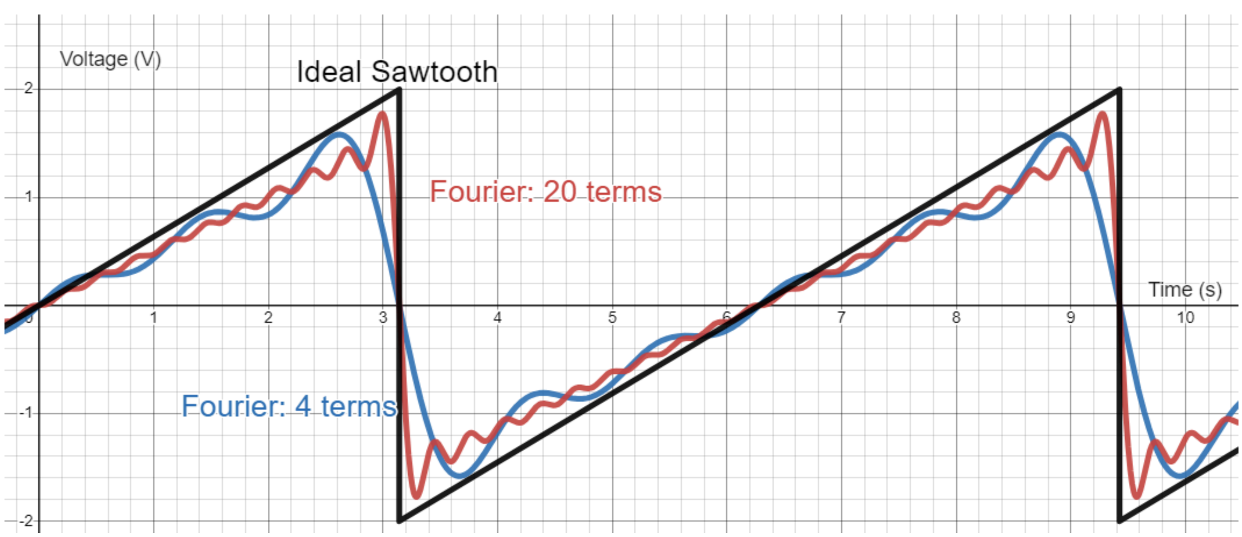


Figure 9: A representation of the first 4 terms (**blue**) and 20 terms (**red**) of the Fourier Series for a Sawtooth wave. Created on Desmos.

It can be seen that more terms more closely approximate the ideal function for the sawtooth wave. While this result is as surprising as the Maclaurin series earlier, the truly revolutionary idea is the Fourier ***transform***.

The function of the sawtooth wave, and its equivalent Fourier ***series***, is the value of the wave at any particular ***time***. The Fourier ***transform***, also a function, is the amplitude of the sine waves that make up the original function at each particular ***frequency*** (Osgood, Stanford Lecture). It represents the same function in a different way: the independent variable on the x-axis is frequency rather than time. The y-axis represents the amplitude of the Fourier series' sine wave of that frequency (Osgood, Stanford Lecture). Figure 10 shows the Fourier transform of the sawtooth wave, and it is simply a series of spikes at each frequency contained in the Fourier series of that wave – each spike equals the coefficient of each term in the series. It is said that the Fourier transform converts a function from the "time domain" to the "frequency domain" (Osgood, Stanford Lecture). This allows scientists to analyze functions in a new and very powerful way.

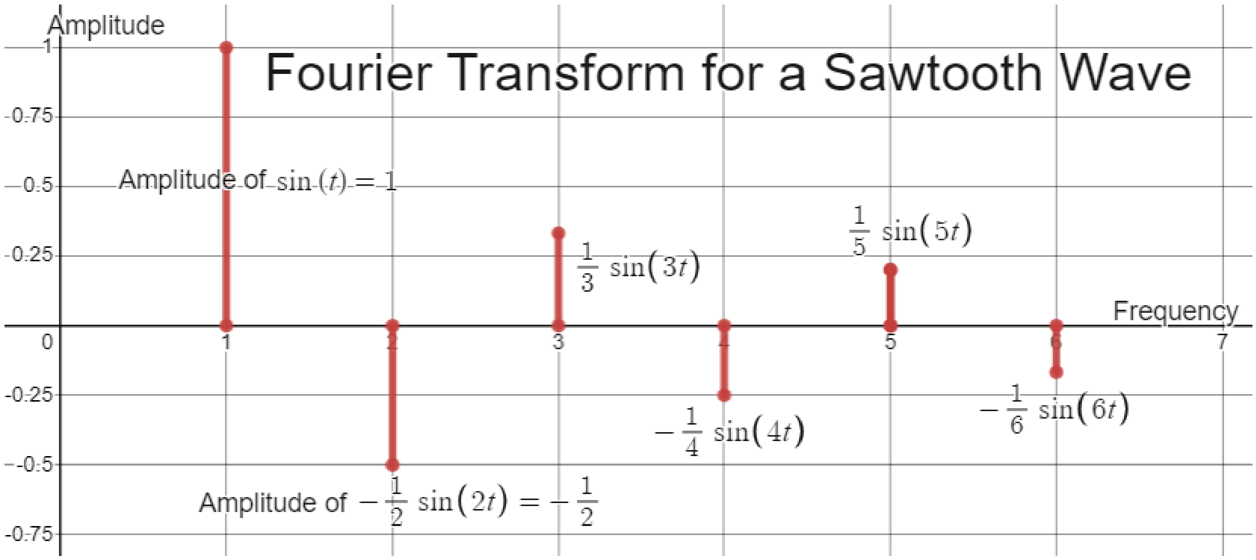

Figure 10: The Fourier transform of the sawtooth wave. Created on Desmos.

What then does the Fourier transform have to do with complex numbers? Here is the equation:

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi ift}\,\mathrm{d}t$$

Created using iMathEQ. Formula (Osgood, Stanford Lecture).

$G(f)$ *is the Fourier Transform of* $g(t)$, $f = frequency$, $t = time$

There is $e^{i\theta}$, and it is now clear, from the earlier analysis of Euler's function, that as $t$ or $f$ increases, this integrand rotates around the complex unit circle. The negative sign indicates it rotates clockwise, and the $2\pi$ simply changes the angular frequency to a traditional frequency: an increase in the product $f \cdot t$ by 1 causes a full rotation. The original function $g(t)$ that is being transformed is in the position of the modulus for a complex number in exponential form; so as the exponential rotates, $g(t)$ is the distance from the origin. For example, consider $g(t) \ = cos(6\pi t) + 1$ from $t = 0$ to $t = 4.5$ (Sanderson, "But What" 2:45). Note that the domain $0 < t < 4.5$ describe a wave that is turned on at 0 and off at 4.5 seconds which is more realistic than an infinitely repeating sinusoid, but also non-standard in a mathematical inquiry. Also note that the +1 is simply to shift the function up so all values are positive for ease of graphing. The integrand, in the complex plane, results in a beautiful spirograph flower (Sanderson, "But What" 2:59 - 6:42) – a "Fourier Flower" if you will – as shown in Figure 11. In many analyses of the Fourier transform, Euler's Formula is used to convert the exponential to sines and cosines, but the deeper understanding of $exp(i\theta)$ gained above allows this more beautiful analysis "in the round".

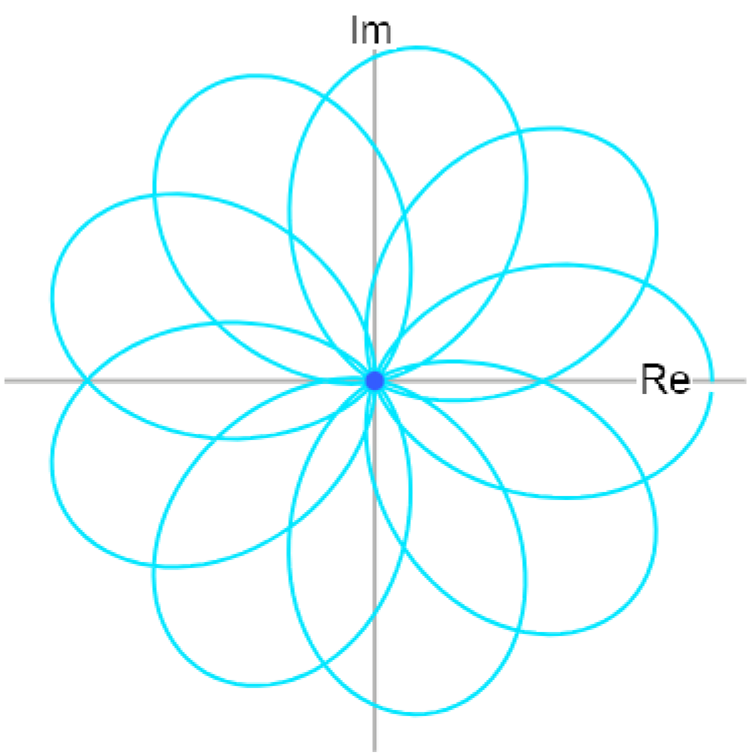


Figure 11: A "Fourier Flower" when $g(t) = cos(6\pi t) + 1$ and $f = 1.33$. Created on Fourier Transform Visualization by Souza.

There are two frequencies involved – the 'actual' frequency of $g(t)$ which is $\frac{6\pi}{2\pi} = 3$ cycles per second (Hz), and the frequency $f$ at which the integral is being evaluated; that is 1.33 Hz in Figure 11. So the last step is to integrate the flower for all $t$ at every value of $f$, resulting in the full Fourier transform $G(f)$. Integrating from -∞ to ∞ is much simplified by the fact that this function is non-zero only when $0 < t < 4.5$, but this limited domain also results in a more interesting Fourier transform: instead of the discrete output of the infinite sawtooth wave, this transform $G(f)$ is a continuous function with values at all $f$. These infinite frequencies are required so they cancel and the function equals zero beyond $0 < t < 4.5$. So integrating this circular function, $g(t)e^{-2\pi i f t}$, is akin to shading in the flower and finding its center – its balance point (Sanderson, "But What" 15:39) if it were a flat, physical flower. The instance of the flower in Figure 11 is symmetrical, and the dot in the center represents the integral – it is zero because all petals are balanced by another with the same area on the opposite side. In fact, the results are quite near the center for most values of $f$, but something interesting happens when the transform is calculated near $f = 3$. As shown in Figure 12, all the petals start to line up to the right and the value of the transform at $f$ (purple dot) has a larger non-zero value. Frequency, $f$, is *near* 3 to show the multiple petals – at *exactly* 3 all the petals overlap exactly and look like a single petal. This lining up of the petals occurs because the rotating exponential 'resonates' with the true frequency of $g(t)$: when $g(t)$ is at a peak, $f \cdot t$ is near a whole number (a full rotation, because of the $2\pi$). So the exponential is to the right, with little or no value to balance it on the left because the smaller values of $g(t)$ occur when $f \cdot t$ is halfway between whole numbers (half a rotation).

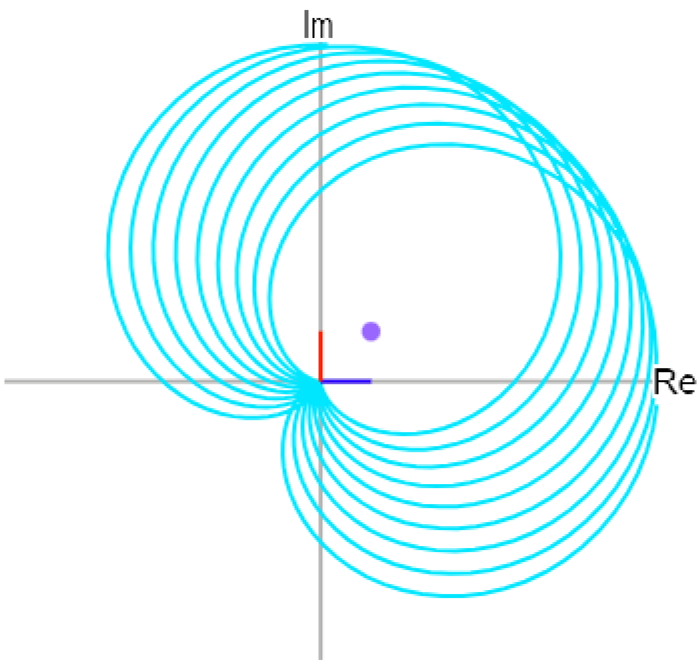


Figure 12: The "Fourier Flower" for $g(t) = cos(6\pi t) + 1$ when $f$ is near 3. Created on Fourier Transform Visualization by Souza.

The full resulting Fourier transform for this simple function is shown in Figure 13. The spike at $f = 0$ is due to the +1 offset, or shift up, in the function (Sanderson, "But What" 7:08). This would be called a DC voltage component if this were an electrical signal. The point marked near $f = 3$ is the frequency that creates the flower in Figure 12.

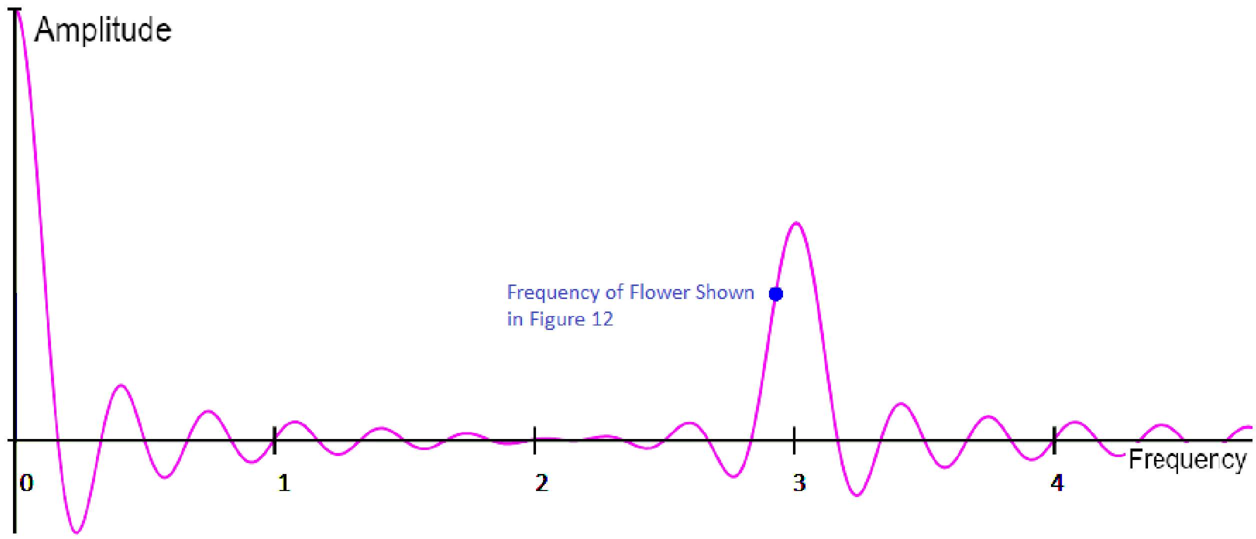


Figure 13: The complete Fourier transform for $g(t) = cos(6\pi t) + 1$. Created on Fourier Transform Visualization by Souza.

To show an example of the utility of the Fourier transform, another look must be taken at the circuit analyzed earlier using phasors. Imagine, now, that the input voltage is connected to an antenna and amplifier so that it captures all passing radio frequencies, instead of being a single constant frequency. The output is the voltage across the capacitor and inductor, which turns the circuit into a band-pass filter. It passes a narrow band of frequencies on to the output at full strength, but attenuates, or weakens, higher and lower frequencies (Kuphaldt Chapter 8). If the capacitance is increased ever so slightly from 1.0 to 1.00917, then this simple filter will ‘tune in’ Planet Radio 100.2 MHz, an FM radio station near Frankfurt, Germany. The Fourier transform of the circuit, also known as the frequency response, is shown in Figure 14.

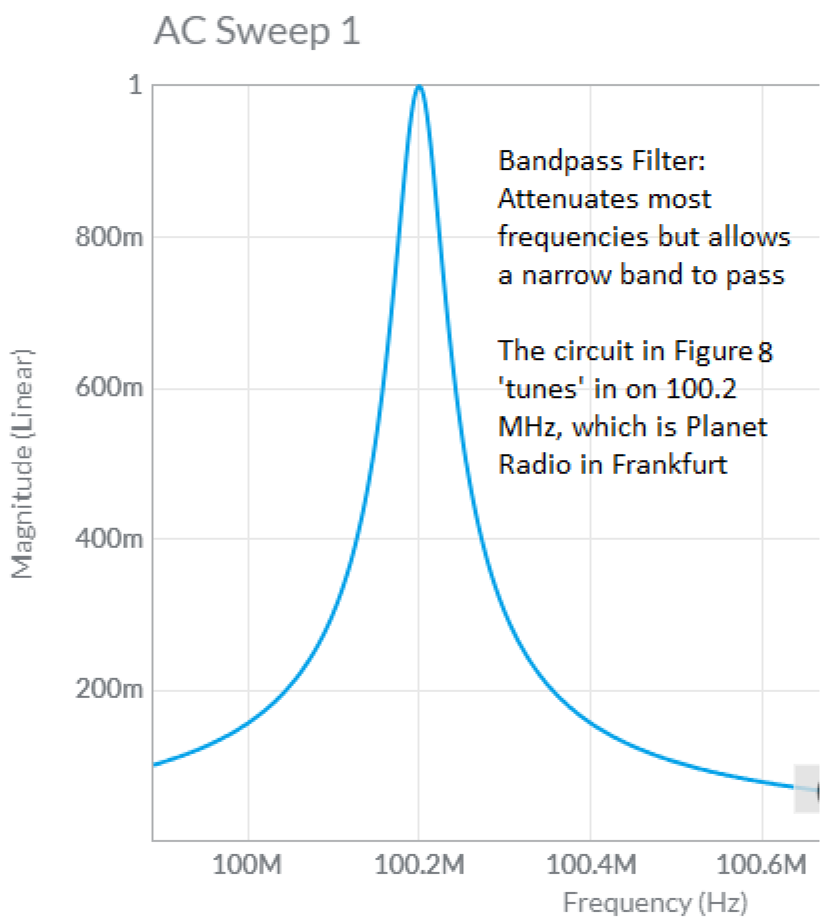


Figure 14: The Fourier Transform (frequency response) of the circuit in Figure 8. Created with Multisim.

This is but a glimpse of the power of the Fourier transform, allowing scientists to understand functions in the frequency domain and providing deeper insight. Want to see exactly how a filter – or any circuit – affects different frequencies? Graph the Fourier Transform. Not only is the Fourier Transform a huge benefit for engineers, it is also another resounding success for $i$ and Euler's Formula, strengthening the case for the reality of complex numbers.

## Schrödinger's Equation

All matter is made of elementary quantum particles, which include quarks, leptons, and bosons, which make up the more-familiar protons and neutrons. An electron is an example of a lepton, and the photon, a particle of light, is a boson (Murayama and Riesselmann). These particles act in non-intuitive ways, but Schrödinger's equation describes them:

$$i\hbar\frac{d}{dt}\Psi = \hat{\mathrm{H}}\Psi$$ (Theoretical and Computational Biophysics Group 53)

This function describes reality at its smallest and most fundamental level, and there is $i$ leading the way. $\hbar$ is simply a constant: Planck's constant $h$ divided by $2\pi$. $\frac{d}{dt}$ and $\hat{\mathrm{H}}$ are both operators: $\frac{d}{dt}$ is the familiar derivative operator and $\hat{\mathrm{H}}$ is the Hamiltonian, an operator that represents the total energy of the system on which it operates, both kinetic and potential (Sherrill). That leaves $\Psi$ which is the quantum wave function, and represents the probability amplitude for a particle; this is the likelihood that the particle is in any particular location. In quantum physics that location can not be known until it is measured and the wave function collapses. $\Psi$ is a complex-valued function, so imaginary numbers are even more deeply intertwined in this equation than the $i$ suggests. Figure 15 is a graphical representation of one wave function, for one particle, that satisfies Schrödinger's equation, and it displays a beautiful spiral shape that has become a recurring theme throughout this analysis. Again, $i$ is found to be an accurate way to represent real-world quantities that are too complicated to be quantified by a single number and have rotating, two-dimensional characteristics.

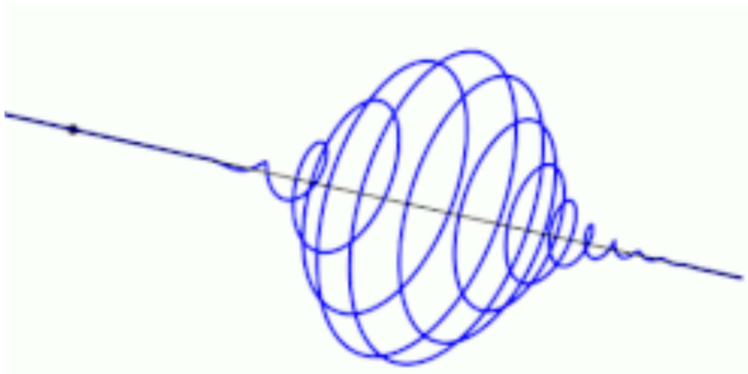

Figure 15: Complex plot of a Gaussian wave function that satisfies Schrödinger's equation ("Gaussian wave").

Though briefly explained here, quantum theory is a colossally complicated subject that is best left to specialists, but what do those specialists say about imaginary numbers? Are imaginary numbers simply a convenience that helps simplify calculations in Quantum Theory, as they are with Steinmetz's method, or are they an indispensable ingredient? Researchers recently investigated this exact question and proposed an experiment involving the transmission of photons that would yield different results for the standard quantum theory using imaginary numbers, and "real quantum theory" which does not (Renou et al.). Three different teams of experimenters performed this experiment, and the results matched those calculated using quantum theory using imaginary numbers (Renou et al.). While further refinement of real quantum theory could possibly account for these differences, current formulations of quantum theory that do not use $i$ do not properly explain experimental evidence.

## **Conclusion**

Imaginary numbers started as a barely-acceptable tool to help solve cubic equations, but have become a fundamental component of some of math's most useful operations. They are necessary when the quantities described are too complicated to be represented in only one dimension and allow the creation of complex numbers which use two. $i$'s ability to rotate a real number out of the real dimension, into the new $i$ dimension and then back, is essential to the rotating nature of $e^{i\theta}$. Euler's Formula reinforces this connection to rotation and circles by equating $e^{i\theta}$ with sine and cosine, two functions that break circles into their two rectangular parts. Steinmetz's phasor algebra, Fourier transforms, and Schrödinger's equation all require this new dimension, and they all model real phenomena. In fact, Schrödinger's equation represents the fundamental nature of matter itself, including every atom in Sally's apples and every real particle in the universe. The answer to the initial research question, then, is that if usefulness in solving real-world problems is our criteria, then imaginary numbers are entirely real. Without them, reality itself can not be properly modeled, so a less accurate name for imaginary numbers can not be imagined.

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